

ELECTION QUOTA

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Я считаю, что совершенно неважно,
кто и как будет в партии голосовать;
но вот что чрезвычайно важно, это –
кто и как будет считать голоса.
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ABSTRACT. We prove that, for the Hamilton's Election Method with the Hare Quota, if, after the "quota assignment," there are unassigned seats, then: (1) At least two parties have positive remainders; (2) There are less unassigned seats than parties with positive remainders.

INTRODUCTION

The following two problems are equivalent:

1. For an election district: How to allocate seats to the party lists in contention "proportionally" to their votes.
2. For a national parliament: How to allocate seats to subnational election districts "proportionally" to their populations.

Among the several methods used, we have the "largest remainder method." In this method, an "election quota" is determined which represents the "price," expressed in either votes, or population, of a seat.

The most natural choice for a quota is the Hare Quota, $q_h := N/S$, where N is either V , the total number of valid votes cast, for Problem 1, or P , the total population of the country, for Problem 2. In both cases, S is the number of seats to be allocated.

We will frame our discussion in terms of Problem 1. The application of our results to Problem 2 is immediate.

Thus, we consider an election district that elects $S > 0$ representatives. Voters vote for party lists and, via the Hamilton Election Method, seats are assigned to the several parties "proportionally" to their votes.

Let us suppose that in a given election, there are $p \geq 1$ parties in contention, and that $v_1, \dots, v_p \geq 0$ denote the valid votes cast for each of the p parties. If V denotes the total number of valid votes cast, then $V = \sum_{i=1}^p v_i$. To avoid trivialities, we assume $p \geq 2$, $\forall i \in \mathbf{p} := \{1, \dots, p\}$,

$v_i > 0$, thus $V > 0$, and $\forall i \in \mathbf{p}$, $v_i < V$.

THE RESULTS

The "Hare Quota" is $q = q_h := V/S > 0$, the "price," expressed in votes, of each seat.

The 1st step, the "quota assignment," assigns to each party as many seats as it can "buy" with the votes it got.

Definition 1. $\forall i \in \mathbf{p}$, party i is assigned $s_i := [v_i/q]$ (the integer part of v_i/q) seats. The "remainders" are

$$\forall i \in \mathbf{p}, r_i := v_i/q - s_i = v_i/q - [v_i/q] \in [0, 1).$$

Proposition. After the 1st step, the number of unassigned seats is $S - \sum_{i=1}^p s_i = \sum_{i=1}^p r_i \geq 0$.

Proof. The result follows from $\sum_{i=1}^p s_i + \sum_{i=1}^p r_i = \sum_{i=1}^p (s_i + r_i) = \sum_{i=1}^p v_i/q = (\sum_{i=1}^p v_i)/q = V/(V/S) = S$. **qed**

N.B. $S, v_1, \dots, v_p, V = \sum_{i=1}^p v_i$ are known integers. Also, $q = V/S$, and $\forall i \in \mathbf{p}$, $v_i/q = v_i/(V/S) = Sv_i/V$, $r_i = v_i/q - [v_i/q]$. Hence, q , and $\forall i \in \mathbf{p}$, $v_i/q, r_i$ are rational numbers known with total precision.

Theorem. If, after the 1st step, there are unassigned seats, i.e., if $S - \sum_{i=1}^p s_i > 0$, then

1. At least two parties have positive remainders.
2. The number of unassigned seats is less than the number of parties with positive remainders

$$S - \sum_{i=1}^p s_i < |\{i \in \mathbf{p} : r_i > 0\}|,$$

where, for any set A , $|A|$ denotes its cardinal.

Proof. Since $\sum_{i=1}^p r_i = S - \sum_{i=1}^p s_i$, if there are unassigned seats, then $S - \sum_{i=1}^p s_i \geq 1 \Rightarrow \sum_{i=1}^p r_i \geq 1$.

1. If only one party has positive remainder, say Party #1, then $1 \leq \sum_{i=1}^p r_i = r_1 \in [0, 1)$, a contradiction.

2. Since $\sum_{i=1}^p r_i \geq 1$, $\exists i \in \mathbf{p}$, $r_i > 0$, and, wlog, $\exists r \in \mathbf{p}$, $\{i \in \mathbf{p} : r_i > 0\} = \{1, \dots, r\}$. Thus $|\{i \in \mathbf{p} : r_i > 0\}| = r$, and $\sum_{i=1}^p r_i = \sum_{i=1}^r r_i$. Now, $\forall i \in \mathbf{p}$, $r_i < 1$, hence $S - \sum_{i=1}^p s_i = \sum_{i=1}^r r_i < \sum_{i=1}^r 1 = r = |\{i \in \mathbf{p} : r_i > 0\}|$. **qed**